Asymptotics for Provisioning Problems of Peering Wireless LANs with a Large Number of Participants*

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Abstract

We consider a model in which wireless LANs are to be provided in a number of locations. The owners of these WLANs have decided to peer with one another so that they can roam in locations other than their own. We address the question of designing a mechanism for deciding the quantities of resources that agents should provide so that the qualities of service are achieved in the locations and a measure of expected welfare is maximized, subject to the mechanism being incentive compatible, rational and feasible (in senses to be described). We show that as the number of participant becomes large there is a limiting problem, whose solution takes a simple form. For instance, it is near optimal to set a fixed fee and allow a participant to use the WLANs in other locations when roaming if he is willing to pay this fee. The advantage of this is that the provisioning policy and fee structure can be easily communicated to the participants.

1 Peer to peer networks of wireless LANs

Access to the Internet is still not as ubiquitous as access to the telephone network. This greatly reduces the economic value of many new portable devices, such as PDAs, tablet computers and smart-phones running the IP protocol. The users of these devices would benefit greatly

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from cost-effective Internet access that is wireless, always-on, ubiquitous and high-speed. However, deploying infrastructure with wide enough coverage to support this is a non-trivial task, especially from the business perspective.

Wireless Local Area Networks (WLANs) are an important developing infrastructure. Specifically, the IEEE 802.11 WLAN standard has grown steadily in popularity since its inception and, at least in metropolitan areas, is now well positioned to complement much more complex and costly technologies such as 3G. This is already happening. WLAN signals of networks set up by individuals for their own use already pervade many cities and such WLAN 'cells' frequently cover greater areas than was originally intended. at their installation. Given how easy it is to gain access to a WLAN once a potential user is within its coverage area, and leaving out the obvious security issues involved, one wonders if individuals could share such infrastructure amongst themselves to achieve ubiquitous Internet access. Sharing comes as a natural idea since WLANs provide large amounts of bandwidth that is mostly underutilized by its local users. Also the pipe that connects the local WLAN users to the Internet is usually of a broadband nature (DSL) and may also be under-used over large time periods. Existing technology allows (or will soon allow) WLAN administrators to control access to their networks and to limit the consumption of network resources by remote (roaming) users.

In this paper we develop an economic model for sharing resources among WLANs. As in existing peer-to-peer (p2p) file sharing systems, such as Gnutella and Kazaa, individual WLAN owners may decide to join the peering group, and abide by certain rules that are imposed by the system in respect of the amount of resources they must contribute. No central entity controls the interaction between the peers, each of whom has full control of his own participation level in the community. Our aim is to optimize these participation rules in order to maximize economic efficiency and reduce free-riding. If there were no such rules, the free riding problem would be debilitating, as each WLAN would offer no resources to others (in order to minimize its own cost, i.e., not decrease the quality of the service provided to its own local customers), while trying to consume as much as it can of the resources of peering WLANs (when roaming in remote areas). Altruism can go some way towards ameliorating this inefficiency, and this may partly explain why existing p2p systems (such as Gnutella and Kazaa) operate with some degree of success, even though studies indicate that the majority of the users are free riders (see Adar and Huberman (2000) and Saroiu et al. (2002)). However, we do not expect that altruism alone can completely correct such inefficiencies. Ubiquitous connectivity is a public good, but We suppose that each peer's preferences (the values he place on connectivity) is known only to himself. The problem is one of 'mechanism design': to determine rules on the peers' contributions and usages so as to maximize social welfare. It is extremely complicated to determine the optimum contribution policy under appropriate informational participation constraints. (the 'second-best' in mechanism design terminology). In general, the contribution required of a peer may depend on the preference declarations of all the other peers and a central planner is required to implement it. Our main result is that as the number of participants in the system becomes large, the optimal contribution policy may be approximated by a simple fixed-fee rule. This may have important practical implications. A peer who wishes to roam must contribute a fixed amount of resources (such as coverage area, the number of roaming peers he accepts in his own WLAN, or a monetary payment). This fee can be computed off-line before the system is instantiated. Note that existing p2p file sharing applications have recently started to incorporate system-specific rules. In most cases these rules are very simple. In Kazaa for example, each peer has an associated 'participation level'. When two peers try contend for downloading a file then the peer with the higher participation level has priority. A peer can increase his participation level by increasing the amount of megabytes that other peers download from him, or by 'integrity-rating' the files he shares.

The idea of creating a p2p system out of WLANs was proposed by Antoniadis et al. (2003b) and (2003a). Their model is different to ours, because they assume that the preference parameters of the participants are known to a global planner. They also discuss security and architectural issues (which may be fundamental for the implementors). We focus on a more demanding, incomplete information model, seeking incentive policies so that participants gain by participating and by being truthful. We also discuss conditions under which no actual money may be part of a peer's contribution, but he may only contribute resources, i.e., make contributions 'in kind'. These issues can be crucial when implementing the system.

We must stress that the business aspect of ubiquitous wireless access is currently receiving lots of attention from communication providers. WISP (Wireless ISP) associations, like *Pass-One*, and large companies, such as *Cometa Networks* (with founders including AT&T, IBM and Intel), are attempting to standardize technologies, protocols and behaviors among existing WISPs in order to make WLAN roaming as seamless as possible. Cometa and other large WISPs attempt to set up new WLAN APs in hot spots and create their own standards, usually by investing a substantial amount of capital in the process. Due to its p2p character, our approach is fundamentally different. The network does not belong to a small number of telecom operators, but to the users themselves.

The paper is organized as follows. In Section 2 we formulate a WLAN peering model and discuss issues of cost. Section 3 formulates the optimization problem. In Section 4 we derive the limiting form amd solution for a large system. Section 5 reports some numerical results.

2 A WLAN peering model

Suppose that a number of WLANs are available in L locations. Each location is a large geographical area like a neighbourhood or a part of a city centre. Potentially n_ℓ WLANs are available in location ℓ. The owners of the WLANs arrange to peer with one another, and thus agent a_{ij}, who is the owner of the jth WLAN in location i, benefits when he roams in other locations.

There are many possible models of benefit and cost. We focus on two. Our first model supposes sparse coverage in all locations, i.e., n_ℓ is much smaller than is needed to cover all of location ℓ. Once a roaming peer is within coverage, he is accepted (perhaps with some fixed probability < 1). So coverage is a public good. The quality of service in location ℓ is defined as the probability Q_ℓ that roaming peer obtains service in location ℓ, and hence is equal to the proportion of area ℓ that is covered. The important issue is to provide incentives for this area to grow, while balancing the resulting costs, assuming that existing WLAN owners can, at some cost, increase their area of coverage (by upgrading or increasing the number of base stations).

Our second model supposes dense coverage, i.e, that locations have enough WLANs to provide full coverage. However, each individual WLAN owner can restrict the number of roaming customers who may simultaneously access the Internet through his infrastructure and so consume some of his bandwidth. The quality of service Q_ℓ now models the geographically averaged probability that a roaming peer is granted service in location ℓ. Now incentives must be given to peers to accept more simultaneous roaming customers, while balancing the resulting opportunity cost of the bandwidth they consume.

We shall focus on the first model. Suppose that agent a_{ij} receives total benefit

$$\theta_{ij} \sum_{\ell=1}^{L} u_{i\ell}(Q_\ell)\,,$$

where the preference parameters $\{\theta_{ij}\}_{j=1}^{n_i}$ are independent, identically distributed realizations of random variables with distribution function F_i. All the F_i are known to all agents, but θ_{ij} is known to agent a_{ij} alone. We allow the possibility that agent a_{ij} may or may not be included in the set of agents who peer with one another, i.e., who share their WLANs, and we denote these possibilities by $\pi_{ij} = 1$ and $\pi_{ij} = 0$ respectively.

The cost of ensuring quality Q_i in location i $c_i(Q_i, \sum_{h,j} \alpha_{hi}\pi_{hj})$, where α_{hi} is the rate at which a single (typical or average) agent from location h generates Internet access requests in location i. It is natural to suppose that $c_i(\cdot,\cdot)$ is increasing in both arguments, since the cost of providing quality level Q_i (by provisioning of bandwidth, availability, etc.) should increase both in Q_i and in the total rate at which agents request Internet access in location i. Showing the functional dependency, the social welfare function is

$$\sum_{i=1}^{L} \sum_{j=1}^{n_i} \pi_{ij}\theta_{ij} \sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) - \sum_{i=1}^{L} c_i \left(Q_i, \textstyle\sum_{h,m} \alpha_{hi}\pi_{hm}\right) .$$

The decision variables π_{ij} and Q_i are to chosen as functions of $\boldsymbol{\theta}$, where this denotes a vector of all the θ_{ij}. Let us specialize here to $c_i = c_i(Q_i)$. We can be generalize this later. Thus the expected social welfare is then

$$\int_{\Theta} \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta})\theta_{ij} \sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i(Q_i(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta}) \,, \tag{1}$$

where Θ is the domain of $\boldsymbol{\theta}$ and $F(\boldsymbol{\theta})$ is its distribution function, i.e., $F(\boldsymbol{\theta}) = \prod_{i,j} F_i(\theta_{ij})$.

Let us make some remarks on the cost. In the traditional model of public good provisioning, $c_i(\cdot)$ is the cost to a central authority of providing the public good. In our model, no central authority exists to provide and manage the WLAN access points. These belong to the peers themselves. Each roaming peer who connects to the Internet through a WLAN consumes some minimum bandwidth. Accepting a roaming customer means allocating an 'WLAN channel'. Every channel that a peer makes available to roaming customers reduces the bandwidth available for his own use. As this is costly, he will control the maximum number of roaming customers to whom he offers connectivity at any point in time.

If a global planner is to ensure a quality of service Q_i then he must supply a total amount of resource (available channels or area of coverage) by extracting it from the existing WLANs in location i. There are two cases to consider. In the first case, the planner acts as a middle man who both pays peers to provide resources and also collect fees. Here, $c_i(\cdot)$ is the cost of the resources that must be purchased in location i. Agent a_{ij} pays $p_{ij}(\boldsymbol{\theta})$, which is his contribution towards the total cost of the services subcontracted by the planner. We require $\sum_{ij} p_{ij}(\boldsymbol{\theta}) \geq \sum_i c_i(\cdot)$.

In the second case, monetary payments are not possible, but only payments in kind. This means that the cost must be measured in the units of the resource that is to be provided. E.g., the cost is linear in the number of channels or percentage of area covered by WLAN services. We must redefine our monetary unit to be a resource unit, and re-scale other functions appropriately. Now $c_i(\cdot)$ is the amount of resource required in location i and $p_{ij}(\boldsymbol{\theta})$ is the amount of resource that agent a_{ij} contributes. For example, we might take $c_i(Q_i) = Q_i$. When we maximize (1) with respect to the $\pi_{ij}(\boldsymbol{\theta})$ and $Q_i(\boldsymbol{\theta})$, it must be subject to L constraints $\sum_j p_{ij}(\boldsymbol{\theta}) \geq c_i(\cdot)$, $i = 1, \ldots, L$.

We must take account of two further constraints. Agent a_{ij} should expect to benefit by participating (individual rationality). He should also have the incentive to report his true value θ_{ij} (incentive compatibility). Let f_i denote the density of F_i, and define

$$g_i(\theta_{ij}) = \theta_{ij} - \frac{1 - F_i(\theta_{ij})}{f_i(\theta_{ij})}. \tag{2}$$

We assume $g_i(\cdot)$ is nondecreasing. In Section 3 we consider the case in which monetary payments are possible and show that we can account for all the above constraints by maximizing (1) subject to

$$\int_\Theta \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) g_i(\theta_{ij}) \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i(Q_i(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta}) \geq 0\,. \tag{3}$$

Let $P(n)$ denote the problem of maximizing (1) subject to (3)[1].

In Section 4 we show that as n becomes large, with $(n_1, \ldots, n_L) = (n\rho_1, \ldots, n\rho_L)$ for some given $\rho_1, \ldots, \rho_L)$, the limiting form of $P(n)$ is $\hat{P}(n)$, defined as: maximize

$$\sum_{i=1}^{L} \left[n_i \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i)\theta_i \, dF_i(\theta_i) - c_i\,(Q_i) \right] \tag{4}$$

with respect to $Q_1, \ldots, Q_L$ and $\pi_1(\cdot), \ldots, \pi_L(\cdot)$, subject to the constraint

$$\sum_{i=1}^{L} \left[n_i \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i) g_i(\theta_i) \, dF_i(\theta_i) - c_i\,(Q_i) \right] \geq 0\,. \tag{5}$$

Suppose $Q_1^*, \ldots, Q_L^*$ and $\pi_1^*(\cdot), \ldots, \pi_L^*(\cdot)$ solve $\hat{P}(n)$. Then $Q_i(\boldsymbol{\theta}) = Q_i^*$ and $\pi_{ij}(\boldsymbol{\theta}) = \pi_i^*(\theta_{ij})$ are feasible for $P(n)$ and maximize (1) to within $o(n)$ of its optimum value.

3 A problem of provisioning WLANs

Internet access in location i is a public good, which is of potential benefit to all agents roaming in location i. As we have said, there are two possibilities. Either there are monetary transfers, so resources can be purchased from those agents based in location i, or there are no monetary transfers, but agents in location i provide resources in exchange for being allowed to roam in

[1]If only payments in kind are possible, then we would need L constraints, equivalent to requiring that each term in the sum on i on the left hand side of (3) is individually nonnegative.

other regions. Throughout what follows we take the first of these viewpoints. Throughout what follows we allow monetary transfers. We also suppose that there is a mechanism for excluding agents from the peering set. Given probability with agent a_{ij} is included. If exclusion is not an option, then we simply make the restriction $\pi_{ij}(\boldsymbol{\theta}) = 1$ for all i, j and $\boldsymbol{\theta}$. We adapt as follows ideas of Hellwig (2003) and Norman (2004).

An allocation is said to be *feasible* if the sum of the payments made by agents in location i covers the cost of providing quality of service Q_i in that location, i.e.,

$$\sum_{i=1}^{L} \left(\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) p_{ij}(\boldsymbol{\theta}) - c_i \left(Q_i(\boldsymbol{\theta}) \right) \right) \geq 0 \tag{6}$$

for all $\boldsymbol{\theta} \in \Theta$. An allocation is *weakly feasible* if the expected sum of the payments covers the expected cost, i.e.,

$$E_{\boldsymbol{\theta}} \left[\sum_{i=1}^{L} \left(\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) p_{ij}(\boldsymbol{\theta}) \ - c_i \left(Q_i(\boldsymbol{\theta}) \right) \right) \right] \geq 0\,. \tag{7}$$

Note that agents who are excluded do not pay.

Suppose agent a_{ij} pays $p_{ij}(\boldsymbol{\theta})$. Let us define

$$V_{ij}(\theta_{ij}) = \int_{\Theta_{-ij}} \pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \textstyle\sum_{\ell} u_i(Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij}))\, dF(\boldsymbol{\theta}_{-ij}) \tag{8}$$

$$P_{ij}(\theta_{ij}) = \int_{\Theta_{-ij}} \pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) p_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij})\, dF(\boldsymbol{\theta}_{-ij})\,. \tag{9}$$

Here $\boldsymbol{\theta}_{-ij}$ denotes the vector of all preference parameters other than that of agent a_{ij}. Its distribution function is $F(\boldsymbol{\theta}_{-ij})$ and its domain is Θ_{-ij}.

Agent a_{ij} must expect to have a positive net benefit and an incentive to report truthfully his value of θ_{ij}. These are the condition of *individual rationality*[2]:

$$\theta_{ij} V_{ij}(\theta_{ij}) - P_{ij}(\theta_{ij}) \geq 0 \tag{10}$$

[2]Since this is a function of an agent's expected benefit, there can be times when he is be required to pay more than he benefits, in which cases he might decide not to participate. Our model make most sense if users make binding agreements to participate, or if there are repeated rounds, so a user who reneges on participating in one round can be punished in subsequent rounds.

and *incentive compatibility*:

$$\theta_{ij}V_{ij}(\theta_{ij}) - P_{ij}(\theta_{ij}) \geq \theta_{ij}V_{ij}(\hat{\theta}_{ij}) - P_{ij}(\hat{\theta}_{ij})\,, \quad \text{for all } \hat{\theta}_{ij} \in [0,1]. \tag{11}$$

We have the following.

Lemma 1 *(a) It is necessary and sufficient for incentive compatibility that (i)* $V_{ij}(\theta_{ij})$ *is nondecreasing in* θ_{ij}*, and (ii)*

$$P_{ij}(\theta_{ij}) = P_{ij}(0) + \theta_{ij}V_{ij}(\theta_{ij}) - \int_0^{\theta_{ij}} V_{ij}(\eta)\,d\eta\,. \tag{12}$$

(b) Given incentive compatibility, a necessary and sufficient condition for individual rationality is $P_i(0) \leq 0$*.*

Proof. Firstly, we must have

$$[\hat{\theta}_{ij}V_{ij}(\hat{\theta}_{ij}) - P_{ij}(\hat{\theta}_{ij})] + [\bar{\theta}_{ij}V_{ij}(\bar{\theta}_{ij}) - P_{ij}(\bar{\theta}_{ij})] \geq [\hat{\theta}_{ij}V_{ij}(\bar{\theta}_{ij}) - P_{ij}(\bar{\theta}_{ij})] + [\bar{\theta}_{ij}V_{ij}(\hat{\theta}_{ij}) - P_{ij}(\hat{\theta}_{ij})]$$

If this were not true then it would be better to declare $\hat{\theta}_{ij}$ when $\theta_{ij} = \bar{\theta}_{ij}$, and/or to declare $\bar{\theta}_{ij}$ when $\theta_{ij} = \hat{\theta}_{ij}$. The above gives $(\hat{\theta}_{ij} - \bar{\theta}_{ij})[V_{ij}(\hat{\theta}_{ij}) - V_{ij}(\bar{\theta}_{ij})] \geq 0$ and hence we find the condition that (i) $V_{ij}(\theta_{ij})$ is nondecreasing in θ_{ij}.

Secondly, since θ_{ij} maximizes $\theta_{ij}V_{ij}(\hat{\theta}_{ij}) - P_{ij}(\hat{\theta}_{ij})$ with respect to $\hat{\theta}_{ij}$, we must have, taking derivatives with respect to θ_{ij},

$$\theta_{ij}V'_{ij}(\theta_{ij}) - P'_{ij}(\theta_{ij}) = 0\,.$$

Integrating the above, we find (12). Thus (i) and (ii) are necessary for incentive compatibility. It is easy to check that they are also sufficient.

Individual rationality is the condition that $\theta_{ij}V_{ij}(\theta_{ij}) - P_{ij}(\theta_{ij}) \geq 0$ for all θ_{ij}. Considering this as $\theta_{ij} \to 0$, we see that individual rationality requires $P_{ij}(0) \leq 0$. Conversely, $P_{ij}(0) \leq 0$ and incentive compatibility, implies individual rationality via (8) and (12). ■

Now consider the problem of maximizing expected social welfare, subject to the constraint that our mechanism is weakly feasible[3], individually rational and incentive compatible. This means we are to maximize

$$\int_\Theta \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta})\theta_{ij} \sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i\left(Q_i(\boldsymbol{\theta})\right) \right] dF(\boldsymbol{\theta})\,. \tag{13}$$

Since the scheme is to be incentive compatible, we can deduce from (12) that the expected sum of the payments in location i is given by

$$\sum_{j=1}^{n_i} \int \pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) p_{ij}(\theta)\, dF(\boldsymbol{\theta}) \tag{14}$$

$$= \sum_{j=1}^{n_i} \int P_{ij}(\theta_{ij})\, dF(\boldsymbol{\theta})$$

$$= \sum_{j=1}^{n_i} P_{ij}(0) + \sum_{j=1}^{n_i} \int \left[\theta_{ij} V_{ij}(\theta_{ij}) - \int_0^{\theta_{ij}} V_{ij}(\eta) d\eta \right] dF(\boldsymbol{\theta})$$

$$= \sum_{j=1}^{n_i} P_{ij}(0) + \sum_{j=1}^{n_i} \int \pi_{ij}(\boldsymbol{\theta}) g(\theta_{ij}) \sum_\ell u_{i\ell}(Q_\ell(\boldsymbol{\theta}))\, dF(\boldsymbol{\theta})\,. \tag{15}$$

Since the scheme is to be weakly feasible, we can use (15) to deduce that our problem is one of maximizing (13) subject to

$$-\sum_{ij} P_{ij}(0) \le -\sum_{ij} \int \left[\pi_{ij}(\boldsymbol{\theta}) g(\theta_{ij}) \sum_\ell u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i\left(Q_i(\boldsymbol{\theta})\right) \right] dF(\boldsymbol{\theta})\,,$$

The maximization is with respect to a choice of the functions $Q_i(\boldsymbol{\theta})$ and the constants $P_{ij}(0)$. Restricting ourself to individually rational payments means we must take $P_{ij}(0) \le 0$ for all i. These enter only through their sum, and we may take every $P_{ij}(0) = 0$. This gives $P_{ij}(\theta_{ij}) \ge 0$ for all θ_{ij}, which is as we wish if the payments are to be made in kind. Hence the problem

[3] In practice, we would like to have the stronger condition of feasibility, so that the required resources to be provided with probability 1. If we are allowed to charge excluded agents, then an argument of Cramton et al. (1987) shows that a scheme which is weakly feasible, incentive compatible and individual rational can always be turned into one that is feasible, incentive compatible and individual rational. See Lemma 5 in the Appendix. However, this requires some monetary transfer payments between the agents, so we are no longer in a market where the only currency is payment in kind. If excluded agents cannot be charged, then it is not yet clear to us whether a similar result can be proved. Perhaps one can only hope for weak feasibility. But the fact that we are providing the required resources on average may be enough. It is possible to modify the optimal weakly feasible scheme so that as $n \to \infty$, with $n_i = n\rho_i$, the probability the scheme is feasible tends to 1 and the percentage reduction in expected social welfare tends to 0.

reduces to one of maximizing (13) subject to

$$\int_{\Theta} \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) g_i(\theta_{ij}) \sum_{\ell} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i\left(Q_i(\boldsymbol{\theta})\right) \right] dF(\boldsymbol{\theta}) \geq 0 \, , \tag{16}$$

The maximum is to be found by pointwise choice of $\pi_{ij}(\boldsymbol{\theta})$ and $Q_i(\boldsymbol{\theta})$. Having found it, we can calculate $V_{ij}(\theta_{ij})$, and then the $P_{ij}(\theta_{ij})$ from (12). Finally, we set $p_{ij}(\boldsymbol{\theta}) = P_{ij}(\theta_{ij})/E_{\boldsymbol{\theta}_{-ij}} \pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ if $\pi_{ij}(\boldsymbol{\theta}) > 0$ and $p_{ij}(\boldsymbol{\theta}) = 0$ if $\pi_{ij}(\boldsymbol{\theta}) = 0$, so that agent a_{ij} pays only if he is included and (9) holds.

We can establish several more important lemmas. Lemma 4 guarantees one of the conditions that we require for incentive compatibility, i.e., (i) on page 9. Lemma 3 is used to prove Theorem 1 and establish the limiting problem $\hat{P}(n)$.

Lemma 2 *If for two agents in location i we have $\theta_{ij} > \theta_{ih}$ an optimal solution must have $\pi_{ij}(\boldsymbol{\theta}) \geq \pi_{ih}(\boldsymbol{\theta})$.*

Proof. If this were not so, consider a new solution, the same as the old, except with the values of $\pi_{ij}(\boldsymbol{\theta})$ and $\pi_{ih}(\boldsymbol{\theta})$ interchanged. This would increase the value of (13) (as $\theta_{ij} > \theta_{ih}$) and not decrease the left hand side of (16) (as $g_i(\theta_{ij}) \geq g_i(\theta_{ih})$). ■

Lemma 3 *There exists a Lagrange multiplier λ such that an optimal solution to $P(n)$ can be found by maximizing The Lagrangian*

$$\int_{\Theta} \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta})(\theta_{ij} + \lambda g_i(\theta_{ij})) \sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - (1+\lambda) c_i(Q_i(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta}) \, . \tag{17}$$

The proof is in the Appendix.

Lemma 4 *$V_{ij}(\theta_{ij})$ is nondecreasing in θ_{ij}.*

Proof. It is sufficient to show that $\pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \sum_{\ell} u_{i\ell}(Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij}))$ is nondecreasing in θ_{ij}. For then integrating with respect to $\boldsymbol{\theta}_{-ij}$ would complete the proof.

So suppose this were not true and that for a fixed $\boldsymbol{\theta}_{-ij}$, and $\theta'_{ij} > \theta_{ij}$ we have

$$\pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \sum_{\ell} u_{i\ell}(Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij})) > \pi_{ij}(\theta'_{ij}, \boldsymbol{\theta}_{-ij}) \sum_{\ell} u_{i\ell}(Q_\ell(\theta'_{ij}, \boldsymbol{\theta}_{-ij}))$$

Denote the integrand in (17) by $s(\boldsymbol{\theta})$ and consider $s(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ and $s(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$. Suppose we make a change in which the values of $Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij})$, $\pi_{k\ell}(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ are interchanged with $Q_\ell(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$, $\pi_{k\ell}(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$ for all k, ℓ. With these changes, denote the integrand by $\bar{s}(\boldsymbol{\theta})$. Then as $\theta_{ij} + \lambda_i g_i(\theta_{ij}) \leq \theta'_{ij} + \lambda_i g_i(\theta'_{ij})$ we find

$$s(\theta_{ij}, \boldsymbol{\theta}_{-ij}) + s(\theta'_{ij}, \boldsymbol{\theta}_{-ij}) < \bar{s}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) + \bar{s}(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$$

This contradicts the fact that the original choices of the $Q_\ell(\boldsymbol{\theta})$, $\pi_{ij}(\boldsymbol{\theta})$ were optimal, since that would require would require $s(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \geq \bar{s}(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ and $s(\theta'_{ij}, \boldsymbol{\theta}_{-ij}) \geq \bar{s}(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$. ■

4 The provisioning problem for a large system

It is difficult to compute and communicate the $\pi_{ij}(\boldsymbol{\theta})$, $Q_i(\boldsymbol{\theta})$ (which maximize (17)) and the payments that the participants are to make. A central authority would have to learn the preference parameters of all the agents and then communicate the required payments to the agents. Fortunately, when n is large the problem becomes easier. Recall that our problem $P(n)$ is to maximize (1) subject to (3). Let us take $n_i = n\rho_i$. We have defined problem $\hat{P}(n)$ as one of maximizing (4) subject to (5). This becomes the problem of maximizing

$$\sum_{i=1}^{L} \left[n_i \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i)\theta_i \, dF_i(\theta_i) - c_i\left(Q_i\right) \right] \tag{18}$$

with respect to $Q_1, \dots, Q_L$ and $\pi_1(\cdot), \dots, \pi_L(\cdot)$, subject to the constraint

$$\sum_{i=1}^{L} \left[n_i \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i) g_i(\theta_i) \, dF_i(\theta_i) - c_i\left(Q_i\right) \right] \geq 0, \tag{19}$$

Theorem 1 *Let Φ_n and $\hat{\Phi}$ denote the optimal values of $P(n)$ and $\hat{P}(n)$ respectively. Then*

$$\hat{\Phi}_n \leq \Phi_n \leq \hat{\Phi}_n + o(n)\,.$$

Moreover, if the decision variables $Q_1^*, \dots, Q_L^*$ and $\pi_1^*(\cdot), \dots, \pi_L^*(\cdot)$ solve $\hat{P}(n)$, then by taking $\pi_{ij}(\boldsymbol{\theta}) = \pi_i^*(\theta_{ij})$ and $Q_i(\boldsymbol{\theta}) = Q_i^*$, for all $\boldsymbol{\theta}$ and i, j, we have a feasible solution for $P(n)$ for which the value of the expected social welfare is $\hat{\Phi}_n$, i.e., suboptimal by only $o(n)$.

Form of the limiting solution

Note that the optimal solution to $\hat{P}(n)$ can be computed off-line. In particular, $\pi_i^*(\theta_i)$ is 1 or 0 as θ_i does or does not exceed a threshold, say θ_i^*. We find that (8) becomes

$$V_i(\theta_{ij}) = \pi_i(\theta_{ij}) \textstyle\sum_\ell u_i(Q_\ell) \tag{20}$$

and from (12) we have

$$P_i(\theta_{ij}) = \theta_i^* V_i(\theta_{ij})\,. \tag{21}$$

Thus every WLAN in location i that is included for roaming is required to pay the same fixed fee $\sum_\ell \theta_i^* u_i(Q_\ell)$. This fee can be calculated (from the n_i, u_i and F_i) and communicated before hearing the values of the preference parameters. $\hat{P}(n)$ is now just

$$\underset{Q_1,\dots,Q_L,\theta_1^*,\dots,\theta_L^*}{\text{maximize}} \quad \sum_{i=1}^{L} \Big[n_i(1 - F_i(\theta_i^*)) \textstyle\sum_\ell u_i(Q_\ell) - c_i(Q_i) \Big] \tag{22}$$

subject to

$$\sum_{i=1}^{L} \Big[n_i(1 - F_i(\theta_i^*))\, \theta_i^* \textstyle\sum_\ell u_i(Q_\ell) - c_i(Q_i) \Big] \geq 0\,. \tag{23}$$

The proof is in the Appendix and is given for the more general case in which only payments in kind are possible. Now we require L constraints

$$\int_\Theta \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) g_i(\theta_{ij}) \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i(Q_i(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta}) \geq 0\,. \tag{24}$$

$i = 1, \dots, L$. However, it now becomes harder to prove the equivalent of Lemma 3. Instead, we make an equivalent assumption.

Assumption 1 *There exist* $\lambda_1, \ldots, \lambda_L > 0$ *such that* $P(n)$ *can be solved by maximizing the Lagrangian*

$$\int_\Theta \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta})(\theta_{ij} + \lambda_i g_i(\theta_{ij})) \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - (1+\lambda_i) c_i \left(Q_i(\boldsymbol{\theta})\right) \right] dF(\boldsymbol{\theta}) \,. \tag{25}$$

The maximization is carried out pointwise. That is, for each $\boldsymbol{\theta}$*, the values of* $\pi_{ij}(\boldsymbol{\theta})$ *and* $Q_i(\boldsymbol{\theta})$ *are chosen to maximize the integrand in* (25).

5 Numerical Results

In this section we will include variations on our model, different costs, the proofs, numerical results for applications to simple problems and further discussion. We are grateful to Panos Antoniadis, Robin Mason, George Polyzos and Ben Strulo for helpful suggestions.

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A Proofs

A.1 Proof of Lemma 3

We will add this proof later. Essentially, it is an application of the intermediate value theorem.

A.2 Proof of Theorem 1

Note that solving $\hat{P}(n)$ is equivalent to solving $P(n)$ under the additional constraints that $\pi_{ij}(\boldsymbol{\theta})$ depends on only θ_{ij} and $Q_i(\cdot)$ depends only on i. This fact immediately gives $\hat{\Phi}_n \leq \Phi_n$. Moreover, if we take as a solution to $P(n)$, $\pi_{ij}(\boldsymbol{\theta}) = \pi_i^*(\theta_{ij})$ and $Q_i(\boldsymbol{\theta}) = Q_i^*$ for all $\boldsymbol{\theta}$ and i, j, then these define a weakly feasible, incentive compatible and individually rational scheme that has expected social welfare equal to $\hat{\Phi}_n$. We can set $p_{ij}(\theta_{ij})$ equal to $P_{ij}(\theta_{ij})$, where $P_{ij}(\theta_{ij})$ is calculated via (8) and (12).

It remains to show that $\Phi_n \leq \hat{\Phi}_n + o(n)$. By Assumption 1, the problem can be solved by maximizing a Lagrangian with Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_L)$. Then for $\bar{\lambda}$ and all other λ we have

$$\Phi = \max_{Q_\ell(\cdot),\pi_{\ell j}(\cdot)} \int_\Theta \sum_{\ell=1}^{L} \left[\sum_{j=1}^{n_\ell} \pi_{\ell j}(\boldsymbol{\theta})(\theta_{\ell j} + \bar{\lambda}_\ell g_\ell(\theta_{\ell j})) \textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h(\boldsymbol{\theta})) - (1+\bar{\lambda}_\ell) c_\ell(Q_\ell(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta})$$

$$\max_{Q_\ell(\cdot),\pi_{\ell j}(\cdot)} \int_\Theta \sum_{\ell=1}^{L} \left[\sum_{j=1}^{n_\ell} \pi_{\ell j}(\boldsymbol{\theta})(\theta_{\ell j} + \lambda_\ell g_\ell(\theta_{\ell j})) \textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h(\boldsymbol{\theta})) - (1+\lambda_\ell) c_\ell(Q_\ell(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta}) \tag{26}$$

We will show that the integral in (26) is bounded above by $\hat{\Phi}_n + o(n)$, where

$$\hat{\Phi}_n = \inf_{\lambda} \max_{Q_\ell,\pi_\ell(\cdot)} \sum_{\ell=1}^{L} n_\ell \left[\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h) \displaystyle\int_0^1 \pi_\ell(\theta_\ell)(\theta_\ell + g_\ell(\theta_\ell))\, dF_\ell(\theta_\ell) - \frac{1}{n_\ell}(1+\lambda_\ell) c_\ell(Q_\ell) \right] \tag{27}$$

Let us now suppose that each F_ℓ is the uniform distribution. It is notationally more elaborate, but routine, to prove the theorem for general F_ℓ.

Imagine dividing the interval $[0,1]$ into k equal parts, defining

$$I_i = \left[\frac{i-1}{k}, \frac{i}{k}\right), \quad i = 1, \ldots, k.$$

Let the random variable $X_{\ell i}$ be the number of the $\theta_{\ell 1}, \ldots, \theta_{\ell n_\ell}$ that are in I_i, Note that $X_{\ell i}$ has a binomial distribution with mean n_ℓ/k, and that by Chebyshev's inequality we have

$$P(|X_{\ell i} - n_\ell/k| > \epsilon) \le \frac{n_\ell(1-1/k)(1/k)}{\epsilon^2} \le \frac{n}{\epsilon^2}\,.$$

We shall use this with $\epsilon = n^{2/3}$. Let us define the set $S = \{\boldsymbol{\theta} : |X_{\ell i} - n_\ell/k| \le n^{2/3}, \text{ for all } \ell, i\}$. Then

$$P(S^c) = P\left(\bigcup_{\ell=1}^{L}\bigcup_{i=1}^{k}\left\{|X_{\ell i} - n_\ell/k| > n^{2/3}\right\}\right) \le \sum_{\ell=1}^{L}\sum_{i=1}^{k} P\left(\left\{|X_{\ell i} - n_\ell/k| > n^{2/3}\right\}\right) \le \frac{1}{n^{1/3}}\,.$$

Let $s(\boldsymbol{\theta})$ denote the integrand in (26). Then we have for (26)

$$\max_{Q_\ell(\cdot),\, \pi_{\ell j}(\cdot)} \int_\Theta s(\boldsymbol{\theta})\, dF(\boldsymbol{\theta}) \le \max_{Q_\ell(\cdot),\, \pi_{\ell j}(\cdot)} \int_S s(\boldsymbol{\theta})\, dF(\boldsymbol{\theta}) + \max_{Q_\ell(\cdot),\, \pi_{\ell j}(\cdot)} \int_{S^c} s(\boldsymbol{\theta})\, dF(\boldsymbol{\theta}) \tag{28}$$

Since $P(S^c) \le 1/n^{1/3}$ we can bound the final term in (28) by $(1/n^{1/3})(nB)$, where B is a bound such that for all $i, j,\ \boldsymbol{\theta}$.

$$\pi_{ij}(\boldsymbol{\theta})(\theta_{ij} + \bar{\lambda}_i g_i(\theta_{ij})) \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - (1+\bar{\lambda}_i)c_i(Q_i(\boldsymbol{\theta})) \le B\,.$$

We bound the first term in on the right hand side of (28) by

$$\max_{Q_\ell(\cdot),\pi_{j\ell}(\cdot)} \int_S s(\boldsymbol{\theta})\, dF(\boldsymbol{\theta}) \tag{29}$$

$$\leq \max_{\substack{Q_1,\dots,Q_L,\theta\in S,\\ \theta_1\in I_1,\dots,\theta_k\in I_k\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^{L}\left[\sum_{i=1}^{k} X_{\ell i}\pi_\ell(\theta_i)(\theta_i+\lambda_\ell g_\ell(\theta_i))\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h) - (1+\lambda_\ell)c_\ell(Q_\ell)\right] \tag{30}$$

$$\leq \max_{\substack{Q_1,\dots,Q_L,\theta\in S,\\ \theta_1\in I_1,\dots,\theta_k\in I_k\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^{L}\left[\sum_{i=1}^{k} (n_\ell/k)\pi_\ell(\theta_i)(\theta_i+\lambda_\ell g_\ell(\theta_i))\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h) - (1+\lambda_\ell)c_\ell(Q_\ell)\right] \tag{31}$$

$$+ B\sum_{\ell=1}^{L}\sum_{i=1}^{k}|X_{\ell i} - n_\ell/k| \tag{32}$$

The second term in (32) is bounded by $n^{2/3}LkB$.

Given any $\epsilon > 0$ we can choose k sufficiently large so that the intervals I_i have very small widths, of $1/k$, and so we can have (using continuity and approximation of an integral by a Riemann sum)

$$\max_{\substack{Q_1,\dots,Q_L,\\ \theta_1\in I_1,\dots,\theta_k\in I_k\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^{L}\left[\sum_{i=1}^{k} (n_\ell/k)\pi_\ell(\theta_i)(\theta_i+\lambda_\ell g_\ell(\theta_i))\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h) - (1+\lambda_\ell)c_\ell(Q_\ell)\right]$$

$$\leq \max_{\substack{Q_1,\dots,Q_L\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^{L} n_\ell\left[\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h)\displaystyle\int_0^1 \pi_\ell(\theta_i)(\theta_\ell+\lambda_\ell g_\ell(\theta_\ell))\, dF_\ell(\theta_i) - (1+\lambda_\ell)c_\ell(Q_\ell)\right] + n\epsilon/2$$

Note that this requires that the term in square brackets be Riemann integrable. I imagine we will want Q restricted to an interval.

Given this k we can then choose n sufficiently large that $n^{2/3}LkB$ is less than $n\epsilon/2$. It follows, that given any $\epsilon > 0$ it is possible to choose k sufficiently large and then n sufficiently large to deduce that for n sufficiently large (but depending on λ),

$$\Phi_n \leq \max_{\substack{Q_1,\dots,Q_\ell\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^{L} n_\ell\left[\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h)\displaystyle\int_0^1 \pi_\ell(\theta_i)(\theta_\ell+\lambda_\ell g_\ell(\theta_\ell))\, dF_\ell(\theta_i) - (1+\lambda_\ell)c_\ell(Q_\ell)\right] + n\epsilon$$

By taking an infimum over λ on the right hand side, assuming the infimum is achieved for some $\lambda = (\lambda_i,\dots,\lambda_L)$ that remain bounded as $n\to\infty$, we deduce $\Phi_n \leq \hat{\Phi}_n + o(n)$. Note that, since we require a bound B that holds for all relevant λ, we want to take this infimum over a bounded set of λ. If the infimum were achieved only as $\lambda \to \infty$ then we could not make this conclusion. ■

Lemma 5 *Let us suppose there is just one location, with $n_1 = n$ agents, none of whom is to be excluded. We take, for notational convenience $c_1(Q) = c(Q)$. If $Q(\cdot)$, $p_1(\cdot), \ldots, p_n(\theta)$ defines an incentive compatible, individually rational and weakly feasible scheme, then there is a new payment function $\hat{p}_i(\cdot)$ such that $Q(\cdot)$, $\hat{p}_1(\cdot), \ldots, \hat{p}_n(\theta)$ is incentive compatible, individually rational and feasible.*

Proof. This is by an argument of Cramton et al. (1987), as follows. Let $E[\,\cdot \mid \theta_i]$ denote expection over $\boldsymbol{\theta}_{-i}$ given θ_i. Recall that $P_i(\theta_i) = E\left[p_i(\boldsymbol{\theta}) \mid \theta_i\right]$. Now define

$$\begin{aligned} \hat{p}_i(\theta) = P_i(\theta_i) &+ \frac{1}{n} c(Q(\theta)) - \frac{1}{n} E\left[c(Q(\boldsymbol{\theta})) \mid \theta_i\right] \\ &- \frac{1}{n-1} \sum_{j \neq i} \Big[P_j(\theta_j) - EP_j(\theta_j)\Big] \\ &+ \frac{1}{n-1} \sum_{j \neq i} \frac{1}{n} \Big[E\left[c(Q(\boldsymbol{\theta})) \mid \theta_j\right] - Ec(Q(\boldsymbol{\theta}))\Big]. \end{aligned} \tag{33}$$

Note that with this definition,

$$\sum_i \hat{p}_i(\theta_i) = c(Q(\theta)) + \sum_j EP_j(\theta_j) - Ec(Q(\boldsymbol{\theta})) \geq c(Q(\theta))$$

so the new payment function is feasible. Also, $\hat{P}_i(\theta_i) = E\left[\hat{p}_i(\boldsymbol{\theta}) \mid \theta_i\right] = P_i(\theta_i)$, so the new payments continue to satisfy the conditions of Lemma 1 and so the scheme is incentive compatible and individually rational. ■

In fact there is an easier proof, with formula more obvious than (33). We will add later this later.